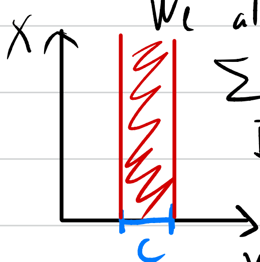
Descriptive Set Theory
Lecture 19

Notation. For a subset A of a Polish space X and Γ a class of sets (e.g. Σ^0_α, Π^0_α, Δ^0_α, $\mathcal{B}$), put
$$\Gamma(X)|_A := \{B \cap A : B \in \Gamma(X)\}.$$

Obs. If $A \subseteq X$ is closed, then $\Pi^0_\alpha(X)|_A \subseteq \Pi^0_\alpha(X)$, where A is treated as a Polish space. However, $\Sigma^0_1(X)|_A \not\subseteq \Sigma_1(X)$ although $\forall \alpha \geq 2$, $\Sigma^0_\alpha(X)|_A \subseteq \Sigma^0_\alpha(X)$.

Cor. For any Polish X, Y where Y is unctbl, $\exists$ Y-universal set $\mathcal{U}$ for $\Sigma^0_\alpha(X)$ and $\Pi^0_\alpha(X)$, $\forall$ ctbl $\alpha \geq 1$.

Proof. By the perfect set property, Y contains a homeom. copy C of $2^{\mathbb{N}}$, hence C is a closed subset of Y.



We already know that $\exists$ C-universal sets for $\Sigma^0_\alpha(X)$ and $\Pi^0_\alpha(X)$, i.e. subsets of $C \times X$. It's enough to find Y-universal sets for the Π-classes, so let $\mathcal{U} \subseteq C \times X$ be C-universal for $\Pi^0_\alpha(X)$. Then $\mathcal{U} \in \Pi^0_\alpha(C \times X)$ $\subseteq \Pi^0_\alpha(Y \times X)$, so $\mathcal{U}$ is Y-universal for $\Pi^0_\alpha(X)$.

by Observation

$\square$

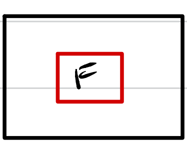
Cor. $\forall$ unctbl Polish X, $\forall \alpha \geq 1$, $\Sigma^0_\alpha(X) \neq \Pi^0_\alpha(X)$, in other words, $\Delta^0_\alpha(X) \subsetneq \begin{matrix} \Sigma^0_\alpha(X) \\ \Pi^0_\alpha(X) \end{matrix} \subsetneq \Delta^0_{\alpha+1}(X)$.

Proof. Let U be a X-universal set for $\Sigma^0_\alpha(X)$. Then $\mathrm{AntiDiag}(U) := \{x \in X : (x,x) \notin U\} = \iota^{-1}(U^c)$, where $\iota : x \mapsto (x,x)$, so $\mathrm{AntiDiag}(U) \in \Pi^0_\alpha(X)$.
But by the Cantor diagonalization, $\mathrm{AntiDiag}(U) \neq U_x$ for each $x \in X$, so it is not Σ^0_α. $\square$

Turning Borel sets into clopen.

Theorem. For any Polish space X and a Borel set $B \subseteq X$, $\exists$ finer Polish top on X in which B is clopen. Moreover, the finer open sets are still Borel in the old Polish topology.

Lemma (Closed into clopen). If $F \subseteq X$ is closed then adjoining F to the top. keeps it Polish.



More precisely, if τ is a Polish top on X and $F \subseteq X$ is τ-closed, then the top. $\mathcal{T}+F$ generated by $\mathcal{T}\cup\{F\}$ is Polish.

Proof. F and F^c are both Polish and $\mathcal{T}+F$ is the top of their disjoint union, which we have proven to be Polish. $\square$

<u>**Lemma (ctbl unions).**</u> Let $\mathcal{T}_n \supseteq \mathcal{T}$ be Polish top. on X containing a Polish top. $\mathcal{T}$ on X. Then the top. $\mathcal{T}'$ generated by $\bigvee_n \mathcal{T}_n$ is still Polish. Moreover, if $\mathcal{T}_n \subseteq \mathcal{B}(\mathcal{T})$, then $\mathcal{B}(\mathcal{T}') = \mathcal{B}(\mathcal{T})$.

(We will see later, by the Luzin–Souslin theorem, that the assumption $\mathcal{T}_n \subseteq \mathcal{B}(\mathcal{T})$ is not needed: if $\mathcal{T}_0 \subseteq \mathcal{T}_1$ are both Polish, then automatically, $\mathcal{T}_1 \subseteq \mathcal{B}(\mathcal{T}_0)$.)

Proof. Let $X' := \prod_n (X, \mathcal{T}_n)$ and let $\iota : X \hookrightarrow X'$ by $x \mapsto (x, x, x, \dots)$. Let $D := \iota(X)$, i.e. the diagonal in $X^{\mathbb{N}}$. X' is a ctbl prod. of Polish top., hence Polish, and $D \subseteq X'$ is closed by the Hausdorffness of X'. Then ι from $(X, \mathcal{T}')$ to D is a homeomorphism, so $\mathcal{T}'$ too is Polish. Indeed, ι is continuous because $\mathrm{proj}_n \circ \iota : (X, \mathcal{T}') \to (X, \mathcal{T}_n)$ is continuous for each n. Also, ι^{-1} is continuous because as a map $\iota^{-1} : D \to (X, \mathcal{T}_n)$

because $\forall\, U \in \mathcal{T}_n$, $(\iota^{-1})^{-1}(U) = \iota(U) = \underbrace{(X \times X \times X \times \overset{\mathcal{T}_n}{\overset{\downarrow}{U}} \times X \times \cdots)}_{\text{open in } X'} \cap D$
so open in D. $\square$

<u>Proof of Borel $\Rightarrow$ clopen.</u> Let $\mathcal{S} \subseteq \mathcal{B}(X)$ be the collection of all Borel sets B for which $\exists$ finer Polish top. on X making B clopen and having the same Borel sets. We have shown that $\mathcal{S}$ contains all open sets by Lemma 1, & $\mathcal{S}$ is closed under ctbl unions by Lemma 2 + Lemma 1 once more. $\square$

<u>Cor.</u> Let $\mathcal{U} \subseteq \mathcal{B}(X)$ be a ctbl collection of Borel subsets of a Polish space X. $\exists$ finer Polish top making each set in $\mathcal{U}$ clopen & having the same Borel sets as the original topology.

Proof. By the theorem, $\exists\, \mathcal{T}_U$ for each $U \in \mathcal{U}$, making U clopen. Then $\bigcup_n \mathcal{T}_n$ generates a Polish top by Lemma 2, in which each $U \in \mathcal{U}$ is clopen. $\square$

<u>Cor.</u> Any Borel function $f: X \to Y$, X, Y Polish, is continuous in a finer Polish top. on X having the

same Borel sets.

Proof. Let $\mathcal{V}$ be a ctbl basis for Y and make all sets in $f^{-1}(\mathcal{V})$ clopen. $\square$

Cor. For any Polish space, $\exists$ 0-dim finer Polish top with the same Borel sets.

Cor. Let $\Gamma \curvearrowright X$ be a Borel action of a ctbl group on a Polish X. $\exists$ finer Polish top on X with the same Borel sets making the action continuous.

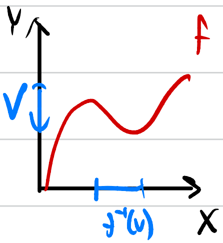
Proof. HW.

Cor. Borel sets have the PSP (perfect set property).

Proof. Any Borel set B is clopen in some finer Polish top $\mathcal{T}'$, while the original Polish top is $\mathcal{T}$. Then by PSP for Polish spaces, there is a embedding $\iota : 2^{\mathbb{N}} \hookrightarrow B$. ι is still continuous with respect to $\mathcal{T}|_B$, hence by the compactness of $2^{\mathbb{N}}$ and Hausdorffness of $\mathcal{T}$, ι is an embedding into $(B, \mathcal{T}|_B)$. $\square$

Making sets open in a new Polish top. for free.

Trivial black magic. Let $(X,\tau_X),(Y,\tau_Y)$ be top. spaces and let $f: X \to Y$ be an arbitrary map. Let τ_X' be coarsest refinement of τ_X that makes f continuous, i.e. $\tau_X' = \tau_X + f^{-1}(\tau_Y)$.



Then on the graph G of f, $G \subseteq X \times Y$, the topologies $\tau_X \times \tau_Y$ and $\tau_X' \times \tau_Y$ coincide.

Proof. Observe that $(f^{-1}(V) \times Y) \cap G = (X \times V) \cap G$, which is open in $\tau_X \times \tau_Y$. $\square$

Obs. Let X, Y be top spaces. A map $f: X \to Y$ is continuous $\Leftrightarrow$ $\pi_f : X \to \mathrm{graph}(f)$ by $x \mapsto (x, f(x))$ is a homeomorphism.

Proof. $\Leftarrow$. If π_f is continuous, then $f = \mathrm{proj}_Y \circ \pi_f$ is continuous.

$\Rightarrow$. If f is continuous, then π_f is cont. hence $\mathrm{id}_X = \mathrm{proj}_X \circ \pi_f$ and $\mathrm{proj}_Y \circ \pi_f = f$ are both continuous. $\pi^{-1} = \mathrm{proj}_X|_{\mathrm{graph}(f)}$ hence continuous. $\square$